Some problems on IVT

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Problems

Problem 1. You decide to estimate e^2 by squaring longer and longer decimal approximations of $e = 2.71828\ldots$.

1. This is a good idea because e is a rational number.
2. This is a good idea because $y = x^2$ is a continuous function.
3. This is a bad idea because e is an irrational number.
4. This is a good idea because $y = e^x$ is a continuous function.

Solution: This is a good idea because $y = x^2$ is a continuous function. By the definition of a continuous function, if you plug in values $x_1, x_2, \ldots, x_n, \ldots$ which get closer to e, the outputs, $x_1^2, x_2^2, \ldots, \ldots x_n^2$, will get closer to e^2.

Problem 2. True or false: You were once exactly 3 feet tall.

Solution: True. When you were born, you were less than 3 feet tall, and now you are more than 3 feet tall. Assuming the height is a continuous function, by IVT, there had to be a point in time when your height was *exactly* 3 feet tall.

Problem 3. True or False: At some time since you were born your weight in pounds equaled your height in inches.

Solution: True. Consider the function of time, t, which is $f(t) = \text{height}(t) - \text{weight}(t)$, the difference between your height in inches at time t and your weight in pound at time t. When you are born, your weight is smaller than height, so $f(0) > 0$. Now, your weight is bigger than your height, and so $f(\text{now}) < 0$. Since we assume f is a continuous function (since it is a difference of continuous functions), by IVT there should be a point in time where $f(t) = 0$.

Problem 4. True or False: Along the Equator, there are two diametrically opposite sites that have exactly the same temperature at the same time.

Solution: True. Define a function on the half of the equator east from Greenwich meridian, as follows. At a point x of the equator, the value $f(x)$ is the difference between the temperatures at x and the point opposite to x. Suppose the value at the point x_0 on the Greenwich meridian is $f(x_0) \neq 0$ (otherwise x_0 would be the point we are looking for). Assume without loss of generality that $f(x) > 0$. Then the value of the point opposite to the point on Greenwich, call it x_1, will be $f(x_1) = -f(x_0) < 0$. Since f is continuous function, there should be a point a where $f(a) = 0$.

Problem 5. Suppose that during half-time at a basketball game the score of the home team was 36 points. **True or False**: There had to be at least one moment in the first half when the home team had exactly 25 points.

Solution: False. Indeed, if only 3-point shots were scored, the team might not have had exactly 25 points. The score at a basketball game is not a continuous function, so IVT does not apply.

Problem 6. True or False: $x^{100} - 9x^2 + 1$ has a root in $[0, 2]$.

Solution: True. If $f(x) = x^{100} - 9x^2 + 1$, we have $f(2) > 0$, $f(1) = 1 - 9 + 1 = -7 < 0$ so by IVT there should be a root of $f(x)$ between 1 and 2. Similarly, $f(1) < 0$ and $f(0) = 1 > 0$ so by IVT there is another root between 0 and 1.